Potential Softening and Eccentricity Dynamics in nearly- Keplerian Discs

Antranik A. Sefilian[1†], Roman R. Rafikov[1,2]

[1] DAMTP, University of Cambridge; [2] IAS, Princeton; [†]aas79@cam.ac.uk

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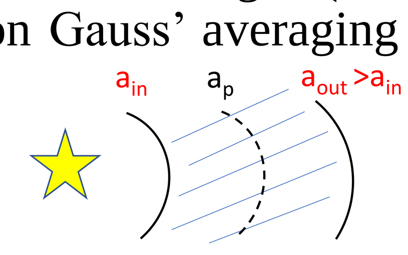
ABSTRACT

In many astrophysical problems involving discs (gaseous or particulate) orbiting a dominant central mass, the gravitational potential of the disc plays an important dynamical role. Its impact on the motion of external objects, as well as on the dynamics of the disc itself, can usually be studied using secular approximation. This is often done by using softened gravity to avoid the singularities that arise when calculating the orbit-averaged potential – disturbing function – of a razor-thin disc using classical Laplace–Lagrange theory. We explore the performance of several softening formalisms proposed in the literature in reproducing the correct eccentricity dynamics in the disc potential. We identify softening models that, in the limit of zero softening, give results converging to the expected behaviour exactly, approximately, or not converging at all. We also develop a general framework for computing the secular disturbing function due to a disc given an arbitrary softening prescription for a rather general form of the interaction potential. Our results demonstrate that numerical treatments of the secular disc dynamics, i.e. by representing the disc as a collection of N gravitationally interacting annuli, are rather demanding: for a given value of the (dimensionless) softening parameter, $\xi \ll 1$, accurate representation of eccentricity dynamics requires $N \sim C\xi^{-\chi}$, with $C \sim O(10)$, $1.5 \lesssim \chi \lesssim 2$. In discs with sharp edges a very small value of the softening parameter ξ ($\lesssim 10^{-3}$) is required to correctly reproduce eccentricity dynamics near the disc boundaries; this finding is relevant for modelling planetary rings.

1. GRAVITATIONAL POTENTIAL OF A DISC

I. Laplace-Lagrange: A common method of computing the orbit-averaged (i.e. secular) gravitational potential R_d due to a disc is based on Gauss' averaging technique:

$$\Rightarrow R_d = \int_{a_{ain}}^{a_{out}} \delta R = n_p a_p^2 \left[\frac{1}{2} A_d e_p^2 + B_d e_p \cos(\varpi_p - \varpi_d)\right]$$



However, this method imposes its own problem: $\mathbf{R_d}$ **is divergent at all locations within the disc** (i.e. $a_{in} \leq a_p \leq a_{out}$). For instance, the expression for the free precession rate induced by the disc would read as:

$$A_d = \frac{\pi G}{2 n_p a_p^2}\left[\int_{a_{in}}^{a_p} \Sigma_d(a)\left(\frac{a}{a_p}\right)^2 b_{3/2}^{(1)}\left(\frac{a}{a_p}\right) da + \int_{a_p}^{a_{out}} \Sigma_d(a)\left(\frac{a_p}{a}\right) b_{3/2}^{(1)}\left(\frac{a_p}{a}\right) da\right]$$

where $b_s^{(m)}(\alpha)$ are the Laplace coefficients: $b_s^{(m)}(\alpha) = \frac{2}{\pi}\int_0^\pi \frac{\cos(m\theta)}{(1+\alpha^2 - 2\alpha\cos\theta)^s} d\theta$

The integrals over $b_{3/2}^{(m)}(\alpha)$ in A_d (as well as B_d) are **singular** – both separately and in their combination – in the vicinity of a test-particle. This is because $b_{3/2}^{(m)} \longrightarrow (1-\alpha)^{-2}$ when $\alpha \to 1$ (i.e. $a_p = a$).

II. Potential Softening: To circumvent this divergence, many authors have resorted to softened inter-particle interactions. This in turn leads to softened Laplace coefficients

$$\mathfrak{B}_s^{(m)}(\alpha, \epsilon) = \frac{2}{\pi}\int_0^\pi \frac{\cos(m\theta)}{[1+\alpha^2 - 2\alpha\cos\theta + \epsilon^2(\alpha)]^s} d\theta$$

where $\epsilon^2(\alpha) > 0$ is the **softening parameter**.

Note: This does not necessarily correspond to substituting $b_s^{(m)}(\alpha)$ with $\mathfrak{B}_s^{(m)}(\alpha,\epsilon)$ in the expressions of A_d and B_d **(Ask for details!)**

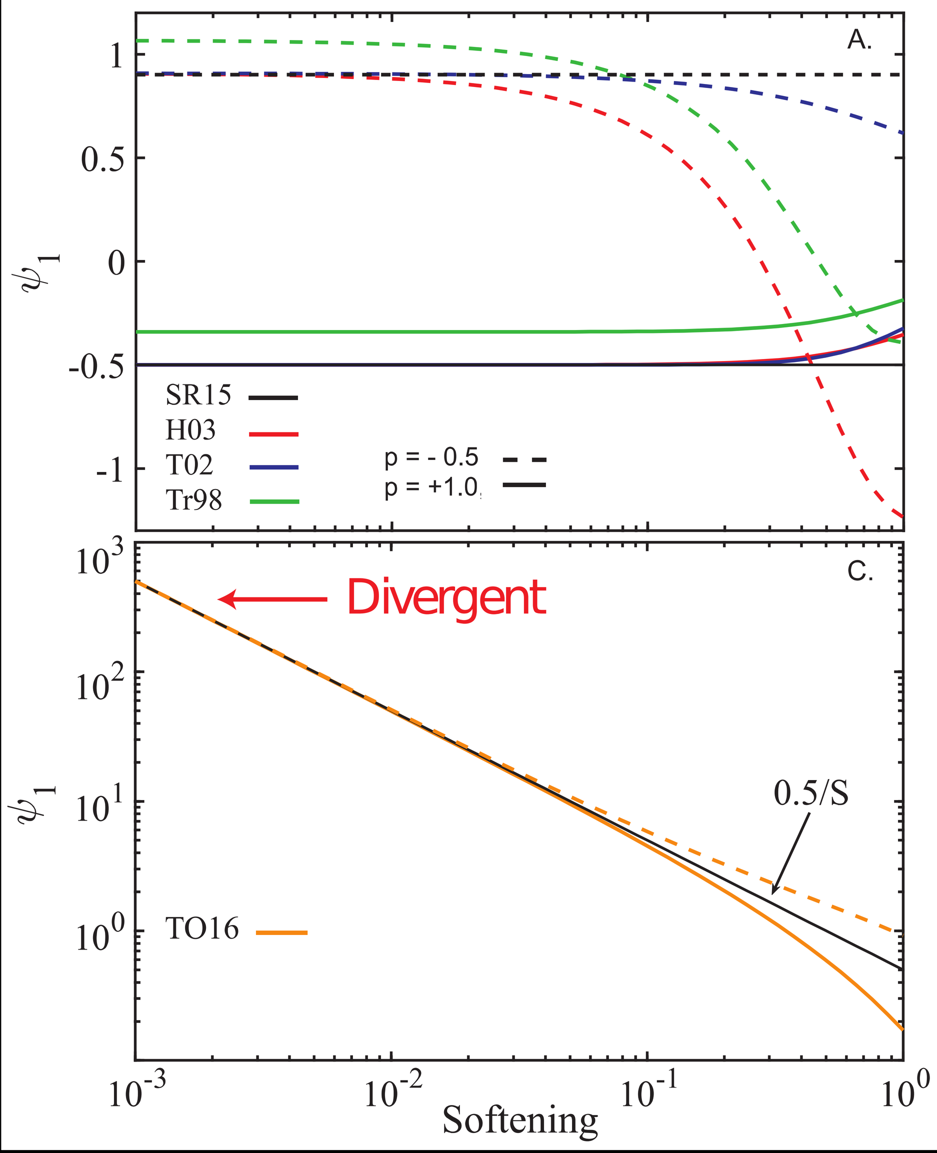
	Tremaine (1998) – Tr98	Touma (2002) – T02	Hahn (2003) – H03	Teyssandier & Ogilvie (2016) – TO16
$\epsilon^2(\alpha)$	β_c^2	$b_c^2/\max(a_p^2, a^2)$	$H^2(1+\alpha^2)$	$S^2\alpha$

III. Heppenheimer's method: The framework first developed by Heppenheimer (1980) – and later extended by many authors [e.g. Silsbee & Rafikov 2015 (**SR15**); Davydenkova & Rafikov 2018 (**DR18**); Sefilian & Touma 2019] – allows the computation of R_d without introducing *ad hoc* softening parameters. Results obtained by this method have been verified against direct orbit integrations.

Objective: Assess how well the different formalisms relying on potential softening reproduce the expected secular dynamics computed using the (unsoftened) Heppenheimer method.

2. DEPENDENCE ON SOFTENING

We consider an axisymmetric power-law (PL) disc with $\Sigma_d(a) = \Sigma_0\left(\frac{a_{out}}{a}\right)^p$ and analyze the behavior of A_d, or equivalently $\psi_1(p) = \frac{n_p a_p}{G\Sigma_d(a_p)}\frac{A_d}{2\pi}$, as a function of softening for the different softening formalisms proposed in the literature.



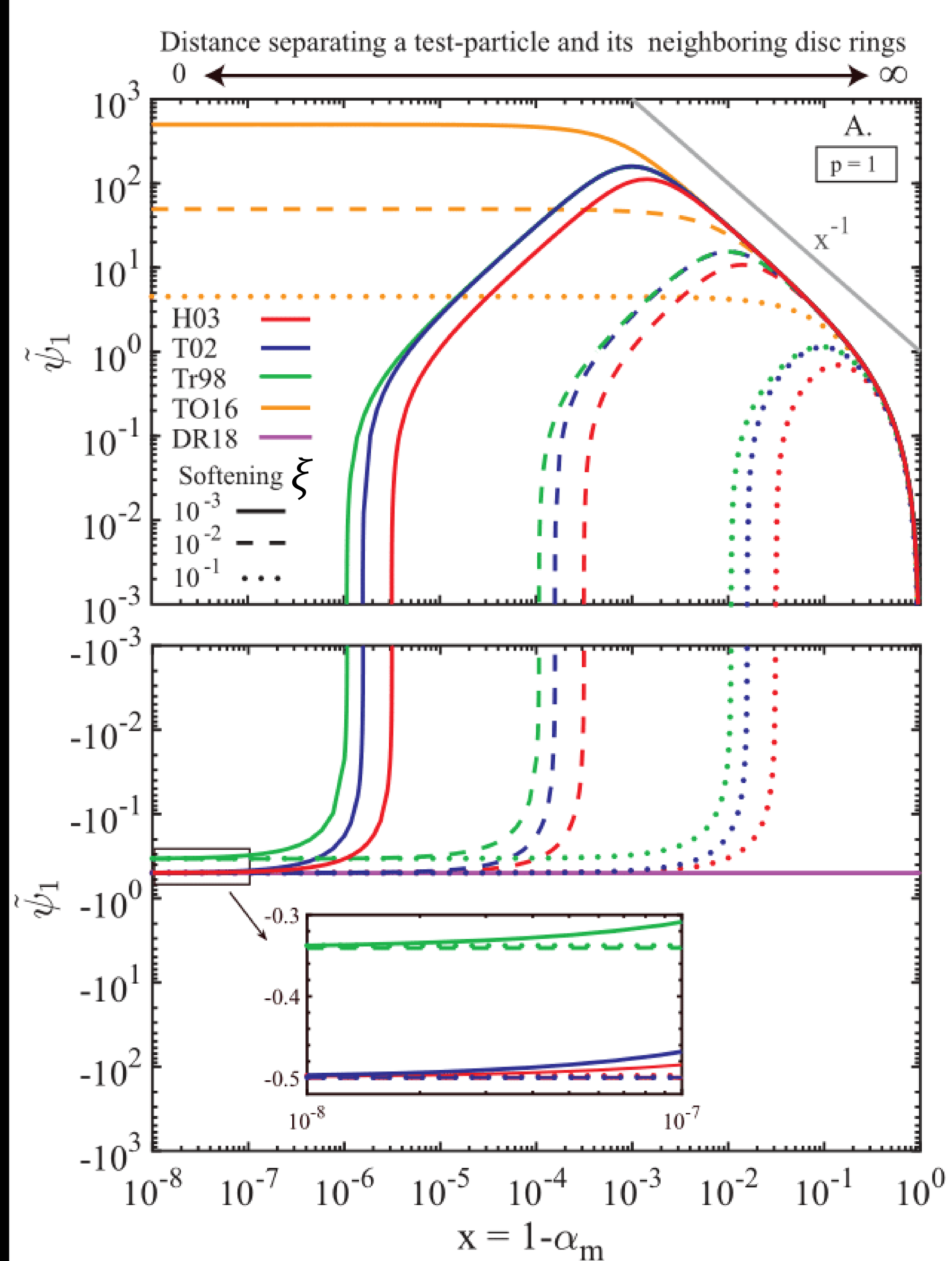
For reference, the expected results as computed by e.g. Silsbee & Rafikov (2015) based on Heppenheimer's method, i.e. without assuming any softening, are shown in black lines.

We find that in the limit of vanishing softening, results obtained by the softening formalism of:

- Touma (2002) and Hahn (2003) converge to the expected results
- Tremaine (1998) show quantitative differences (~20-30%)
- Teyssandier & Ogilvie (2016) are not convergent (like the classical Laplace-Lagrange approach)

We find similar behavior for both eccentric, non-axisymmetric PL discs and non-PL discs (e.g. Gaussian rings)

3. DETAILS OF CONVERGENCE (or not)



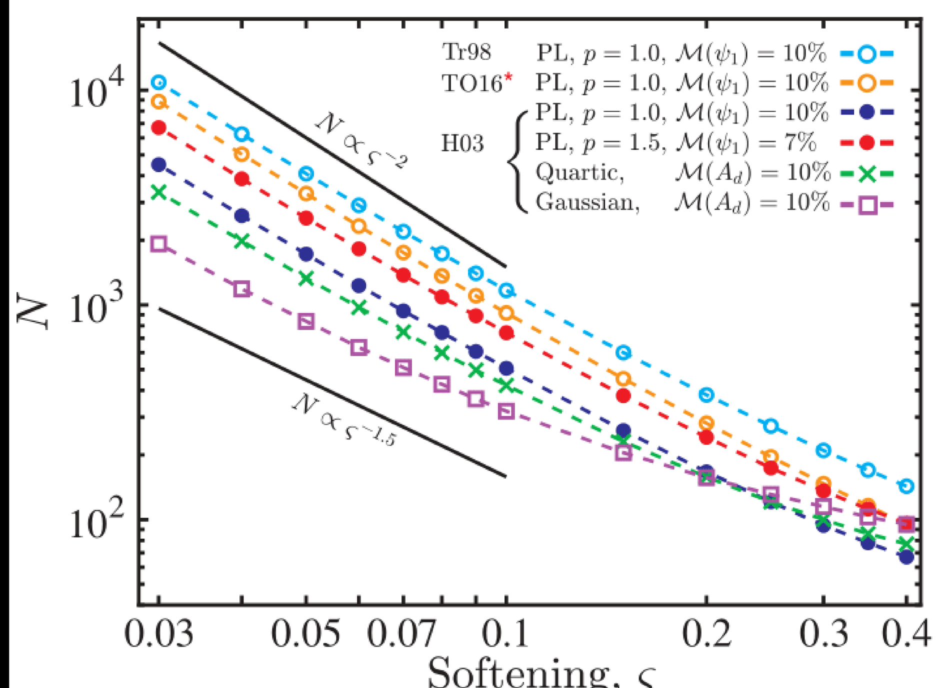
Behavior of the axisymmetric component of R_d due to a PL disc (with p = 1) as a function of the relative separation between a given test-particle orbit and the nearest neighboring disc rings.

→ Secular dynamics in **softened** power-law discs is dictated by the delicate balance between the opposing contributions of the disc rings that are close to (i.e. with $x \lesssim \xi$, negative) and distant (i.e. with $x \gtrsim \xi$, positive) from a given particle orbit.

→ The softening model of TO16 yields inaccurate (divergent) results due to its inability to capture the dynamical effects of disc rings that are adjacent to the test-particle orbit (those with $x \lesssim \xi$)

Similar results are obtained for eccentric discs.

4. IMPLICATIONS FOR NUMERICAL APPLICATIONS



Scaling of number of softened disc annuli (rings) N with the softening parameter ς to ensure convergence of disc-driven free precession A_d in numerically discretized discs to the expected results in *continuous* softened discs.

In numerical studies, discs are often treated as a collection of N softened annuli or rings with prescribed spacing interacting gravitationally with each other.

→ A fine numerical sampling, with $N \sim C\varsigma^{-\chi}$ (such that $C \sim O(10)$ and $1.5 \lesssim \chi \lesssim 2$) is required to ensure that the correct secular behavior is reproduced

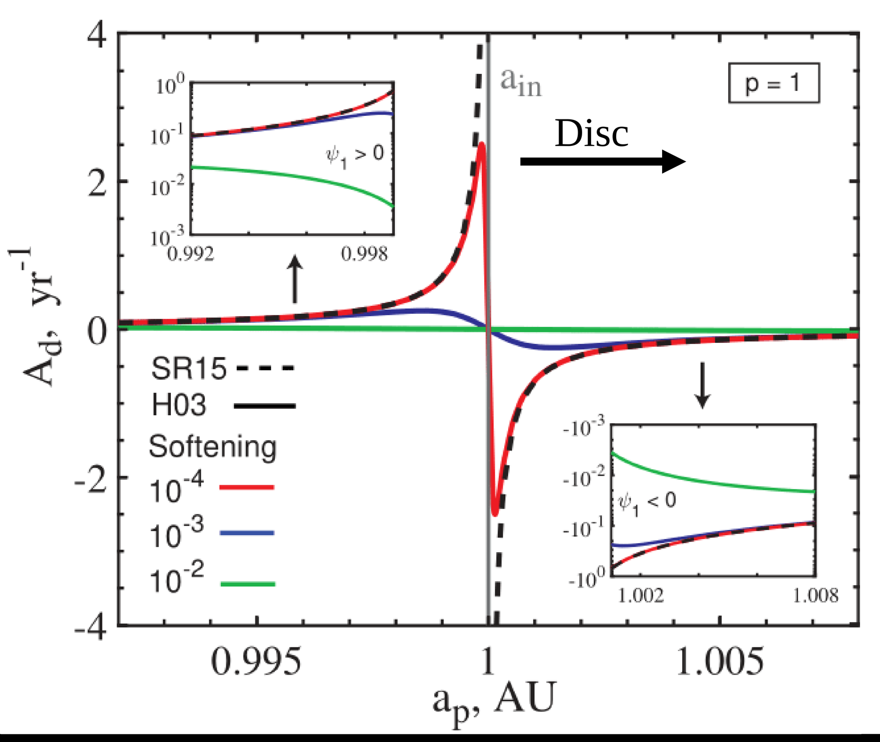
→ This could make numerical studies challenging!

Similar results are obtained for eccentric discs.

Accurately capturing the secular dynamics of particle orbits near the sharp edges of discs/rings requires using very small values of softening

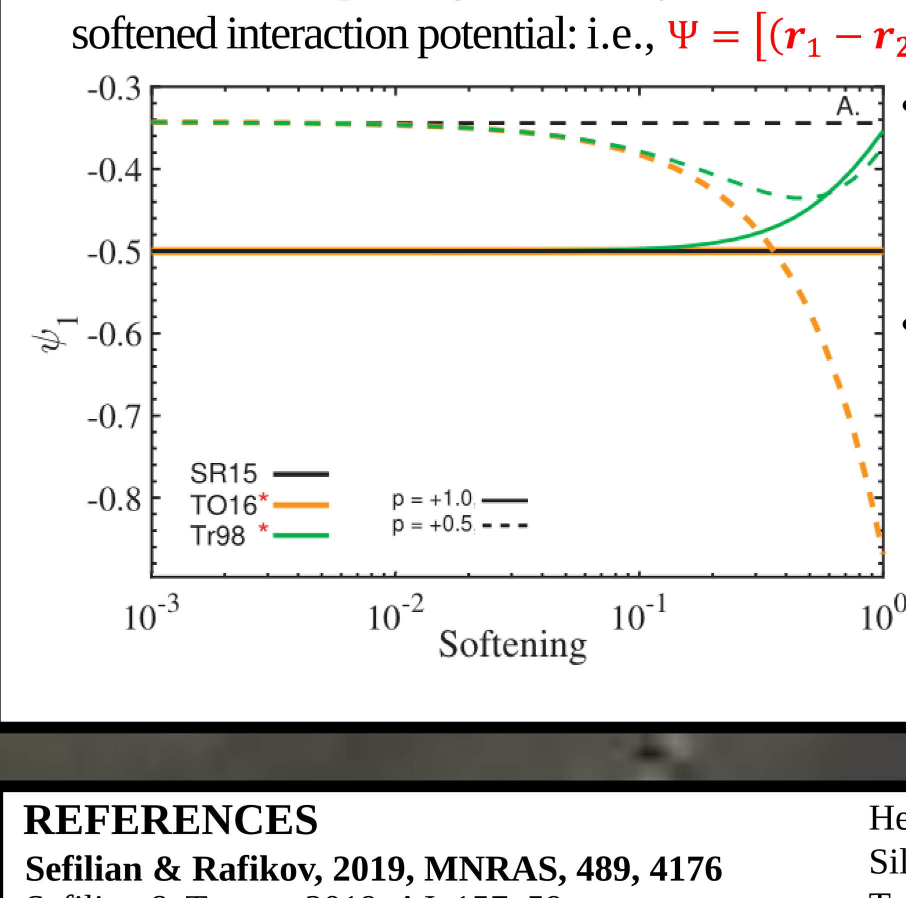
→ Problematic e.g. for numerical modelling of planetary rings.

Similar results are obtained for (i) eccentric discs, and (ii) other softening formalisms



5. GENERAL FRAMEWORK FOR COMPUTING R_d

- We also developed a general analytical framework for computing R_d given an arbitrarily softened interaction potential: i.e., $\Psi = [(r_1 - r_2)^2 + \mathcal{F}(r_1, r_2)]^{-1/2}$ (see Appendix A in here)



- We can recover the expressions of both Touma (2002) and Hahn (2003) if we set $\mathcal{F}(r_1 r_2) = b_c^2 = cte$ and $\mathcal{F}(r_1 r_2) = H^2(r_1^2 + r_2^2)$, respectively.
- The figure shows that an accurate implementation of the softened potential suggested in both Tremaine (1998) and Teyssandier & Ogilvie (2016) leads to the recovery of the expected dynamical behavior in the limit of small softening.

Ask for details!

SUMMARY

- In the limit of zero softening,
  - The softening methods of both Touma (2002) and Hahn (2003) correctly reproduce the expected eccentricity dynamics in razor-thin discs.
  - The softening method of Tremaine (1998) yields convergent results. However, quantitative differences (of up to ~20-30%) are observed.
  - The softening method as implemented by Teyssandier & Ogilvie (2016) does not result in convergent results.
- Numerical studies of secular eccentricity dynamics in softened discs must obey important constraints (number of rings, magnitude of softening, etc..)
- A direct replacement of the classical Laplace coefficients with their softened analogues is not sufficient and justified.

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